A practical walk through formal scattering theory

Connecting bound states, resonances, and scattering states in exotic nuclei and beyond

The Lippmann-Schwinger equation

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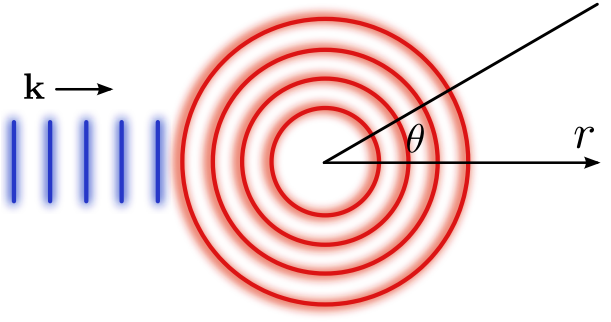
Scattering setup

- we consider two particles with masses m_1 and m_2 at positions $\mathbf{r}_1$ and $\mathbf{r}_2$
- we assume that the interaction does not depend on absolute particle positions
- then we can **neglect the overall center-of-mass motion** and work only with the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and reduced mass $\mu = m_1 m_2/(m_1 + m_2)$
- for the two particles scattering off one another, we physically expect that the wavefunction describing their relative motion is given as a sum of an incoming plane wave and an outgoing spherical scattered wave:

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{r}) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + f_k(\theta)\frac{\mathrm{e}^{ikr}}{r} \tag{1}$$



- all physics information is contained in the scattering amplitude $f_k(\theta)$

Free Hamiltonian

- the **free Hamiltonian** H_0 is an important reference operator
  - differential operator ($\sim \mathrm{d}^2/\mathrm{d}r^2$) in configuration space
  - diagonal multiplicative operator in momentum space
- it has a purely continuous spectrum of plane-wave states:

$$H_0|\mathbf{k}\rangle = \frac{\mathbf{k}^2}{2\mu}|\mathbf{k}\rangle \tag{2}$$

  - this is for a two-body system with relative momentum $\mathbf{k}$ and reduced mass μ

Note

- the plane-wave states $\mathbf{k}$ are not elements of the physical Hilbert space
  - $\langle \mathbf{r}|\mathbf{k}\rangle = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$ is not a normalizable wavefunction ($\notin L^2(\mathbb{R}^3)$)
- strictly one should work with wave packets $|\phi\rangle \sim \int \mathrm{d}^3p\, g(\mathbf{p})|\mathbf{p}\rangle$
- nevertheless it is convenient and permissible to work with plane-wave states because every element of the Hilbert space can be expanded in them (Fourier transform!)

Interaction potential

- we will consider generic potentials V written as operators
- we assume these potentials to be time independent, but they may in principle depend on energy, $V = V(E)$
- taking matrix elements gives a concrete representation, for example:

$$V(\mathbf{r}', \mathbf{r}) = \langle \mathbf{r}' | V | \mathbf{r} \rangle$$

- the most intuitive (and familiar) potentials are local and spherically symmetric in coordinate space: $V(\mathbf{r}', \mathbf{r}) = V(r)\delta^{(3)}(\mathbf{r} - \mathbf{r}')$, with $r = |\mathbf{r}|$

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- in the following, we will work in **momentum space**
- then $V(\mathbf{p}', \mathbf{p}) = \langle \mathbf{p}' | V | \mathbf{p} \rangle$ is related to $V(\mathbf{r}', \mathbf{r})$ by Fourier transformation
- we typically project onto fixed partial waves, denoted by l
- for local potentials, this leads to the following expression:

$$V_l(p', p) = 4\pi \int_0^\infty \mathrm{d}r \, r^2 j_l(p'r) V(r) j_l(pr) \,, \tag{3}$$

where $j_l(z)$ it the l-th spherical Bessel function

Partial-wave projected potentials

- numerically, it is better to use Riccati-Bessel functions $\hat{j}_l(z) = z j_l(z)$
- this explicitly reflects the cancellation of the singularity at $r = 0$

$$V_l(p', p) = \frac{4\pi}{p'p} \int_0^\infty \mathrm{d}r \, \hat{j}_l(p'r) V(r) \hat{j}_l(pr) \tag{4}$$

Python implementation

```python
# lib/potential.py

from scipy.integrate import quad

class LocalPotential(Potential):
  # [...]

  # Specific subclasses implement __call__(self, r)

  def get(self, ell, p, q):
    return 4.0 * np.pi / (q * p) * quad( \
      lambda r: riccati_j(ell, p * r) * self(r) * riccati_j(ell, q * r), \
      0.0, np.inf \
    )[0]
```

Plane-wave boundary condition

- consider the stationary Schrödinger equation: $H|\psi\rangle = E|\psi\rangle$ with $H = H_0 + V$
- this equation alone does not specify a boundary condtion for solutions $\psi(\mathbf{r}) = \langle \mathbf{r}|\psi\rangle$
- a scattering state should be such that for $V \to 0$, $|\psi\rangle \to |\mathbf{k}\rangle$
- moreover, the state should be one that evolved from a free state in the infinite past

Interlude: time dependence

- the full Schrödinger equation is $\mathrm{i}\dfrac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle = (H_0 + V)|\psi(t)\rangle$

- Green's functions solve the time dependence:

$$G_0(t) = G_0^{(+)}(t) = \begin{cases} -\mathrm{i}\mathrm{e}^{-\mathrm{i}H_0 t} & t > 0\,, \\ 0 & t \leq 0 \end{cases} \tag{5}$$

  - this is the free retarded Green's function
  - advanced Green's functions vanish for $t \geq 0$
  - full Green's functions are defined analogously with $H_0 \to H$ in Eq. (5)
  - sometimes these operators are denoted by $U_0(t)$ and $U(t)$

- time dependence of waves is conventionally written $\sim \mathrm{e}^{-\mathrm{i}Et}$
  $\hookrightarrow$ $\mathrm{e}^{\pm\mathrm{i}kr}$ represent in- and outgoing spherical waves, respectively

- $G_0(z)$ is the Fourier tranform $G_0(E)$ of $G_0(t)$, analytically continued into the whole complex energy plane

Interlude: adiabatic switching

- define a free **in state** that was equal to the exact interacting state in the infinite past:

$$|\psi_{\text{in}}(t)\rangle = \lim_{t' \to -\infty} \mathrm{i} G_0^{(+)}(t - t')|\psi(t')\rangle \tag{6}$$

- this allows us to write down a formal solution using the full Green's function:

$$|\psi(t)\rangle = |\psi_{\text{in}}(t)\rangle + \lim_{\varepsilon \to 0^+} \int \mathrm{d}t\, \mathrm{e}^{-\varepsilon t'} G^{(+)}(t - t') V |\psi_{\text{in}}(t)\rangle \tag{7}$$

- if the states monochromatic (fixed definite energy), the damping factor $\mathrm{e}^{-\varepsilon t'}$ with $\varepsilon \to 0^+$ is required in Eq. (7)
- approximately this can be interpreted as multiplying the potential V with $\mathrm{e}^{-\varepsilon t'}$, adibatically switching it off in the infinite past
- when wave packets are used to represent proper physical scattering states, such damping factors are not necessary; their presence otherwise permits the convenience of working with simple plane-wave states

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- both conditions can be enforced with the ansatz

$$|\psi_{\mathbf{k}}^{(+)}\rangle = |\mathbf{k}\rangle + (E_{\mathbf{k}} - H_0 + \mathrm{i}\varepsilon)^{-1} V |\psi_{\mathbf{k}}^{(+)}\rangle \tag{8}$$

  with $E_{\mathbf{k}} = p^2/(2\mu)$
- this is the **Lippmann-Schwinger equation** for the scattering state $|\psi_{\mathbf{k}}^{(+)}\rangle$

Notes

- free Green's function $G_0(z) = (z - H_0)^{-1}$ appears in Eq. (8) with $z = E_{\mathbf{k}} + \mathrm{i}\varepsilon$
- $\varepsilon \to 0$ is implied in all equations, this implements the adiabatic switching
- linear equation becomes an integral equation when projected onto a concrete representation

The Lippmann-Schwinger equation

- consider further Eq. (8) and apply V from the left:

$$V|\psi_{\mathbf{k}}^{(+)}\rangle = V|\mathbf{k}\rangle + VG_0(E_{\mathbf{k}} + \mathrm{i}\varepsilon)V|\psi_{\mathbf{k}}^{(+)}\rangle \tag{9}$$

- define an operator T via $V|\psi_{\mathbf{k}}^{(+)}\rangle = T|\mathbf{k}\rangle$
  - for now, this merely assumes the existence of such an operator
  - it will be justified later
- using the definition of T, we can write:

$$T|\mathbf{k}\rangle = V|\mathbf{k}\rangle + VG_0(E_{\mathbf{k}} + \mathrm{i}\varepsilon)T|\mathbf{k}\rangle \tag{10}$$

- this represents the **Lippmann-Schwinger equation** for the operator T

Notes

- since $\mathbf{k}$ is arbitrary in Eq. (10), we postulate at the operator level: $T = V + VG_0T$
- $T = T(E_{\mathbf{k}} + \mathrm{i}\varepsilon)$ carries an implicit energy dependence via G_0
- alternative form: $T = V + TG_0V$ (seen to be equivalent by iteration)

The T-matrix

- in practice, we want an explicit representation of T
- we apply $\langle \mathbf{p}|$ from the left and define $T(E_\mathbf{k}+\mathrm{i}\varepsilon;\mathbf{p},\mathbf{k}) = \langle \mathbf{p}|T(E_\mathbf{k}+\mathrm{i}\varepsilon)|\mathbf{k}\rangle$
- this is called the **T-matrix**, and it satisfies

$$\langle \mathbf{p}|T|\mathbf{k}\rangle = \langle \mathbf{p}|V|\mathbf{k}\rangle + \langle \mathbf{p}|VG_0(E_\mathbf{k}+\mathrm{i}\varepsilon)T|\mathbf{k}\rangle \tag{11}$$

- Eq. (11) involves the momentum-space potential $V(\mathbf{p},\mathbf{k}) = \langle \mathbf{p}|V|\mathbf{k}\rangle$
- to solve Eq. (11), we need to fully write out the second term on the right
  - insert complete sets of momentum states
  - note that Green's function is diagonal: $\langle \mathbf{q}|G_0(z)|\mathbf{q}'\rangle = G_0(z;\mathbf{q})\delta^{(3)}(\mathbf{q}-\mathbf{q}')$

$$T(E_\mathbf{k}+\mathrm{i}\varepsilon;\mathbf{p},\mathbf{k}) = V(\mathbf{p},\mathbf{k}) + \int \frac{\mathrm{d}^3q}{(2\pi)^3} V(\mathbf{p},\mathbf{q})G_0(E_\mathbf{k}+\mathrm{i}\varepsilon;\mathbf{q})T(E_\mathbf{k}+\mathrm{i}\varepsilon;\mathbf{q},\mathbf{k}) \tag{12}$$

- this integral equation can be solved numerically via **discretization**

Momentum discretization

Numerical quadrature

- a quadrature rule is a set of mesh points p_i together with associated weights w_i such that for a function $f(p)$ it holds that

$$\int_a^b f(p)\,\mathrm{d}p \approx \sum_{i=1}^{N} w_i f(p_i) \tag{13}$$

  - the interval boundaries may be among the p_i (closed rule) or not (open rule)
  - increasing N improves the approximation in Eq. (13)

Mesh class

- a very common choice is **Gauss-Legendre quadrature**

```
from lib.mesh import *
mesh = GaulegMesh(16, 0.0, 1.0)
print(mesh.ps())
print(mesh.ws())
```

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Numerical integration

On-shell singularity

- consider now Eq. (10) written out and with explicit expression for G_0
- assume furthermore we have projected the equation on a single fixed partial wave:

$$T(E_k + \mathrm{i}\varepsilon; p, k) = V(p, k) + \int_0^\Lambda \frac{\mathrm{d}q\, q^2}{2\pi^2} \frac{V(p, q) T(E_k + \mathrm{i}\varepsilon; q, k)}{k^2 + \mathrm{i}\varepsilon - q^2} \tag{14}$$

- for convenience, we have introduced a momentum cutoff Λ
- for $\varepsilon = 0$, we would be integrating over the singularity at $q = p$, which can be isolated by writing

$$k^2 + \mathrm{i}\varepsilon - q^2 = -(q - k - \mathrm{i}\tilde{\varepsilon})(q + k)\,, \tag{15}$$

  where $\lim_{\tilde{\varepsilon}\to 0}$ is equivalent to $\lim_{\varepsilon \to 0}$
- this can now be dealt with using the **principal value formula**

$$\lim_{\varepsilon \to 0} \frac{1}{x \pm \mathrm{i}\varepsilon} = \mathrm{PV}\frac{1}{x} \mp \mathrm{i}\pi\delta(x) \tag{16}$$

- but how do we realize this with a numerical quadrature?

Numerical principal value integration

- schematically we are dealing with an integral of the form

$$\lim_{\varepsilon\to 0}\int_0^\Lambda \mathrm{d}q\,\frac{f(q)}{q-(k+\mathrm{i}\varepsilon)} = \mathrm{PV}\int_0^\Lambda \mathrm{d}q\,\frac{f(q)}{q-k} + \mathrm{i}\pi f(k) \tag{17}$$

  where we have defined

$$f(q) = -\frac{q^2}{2\pi^2}\frac{V(p,q)T(E_k;q,k)}{q+k} \tag{18}$$

- now, we add $0 = f(k) - f(k)$ in Eq. (17) and get:

$$\mathrm{PV}\int_0^\Lambda \mathrm{d}q\,\frac{f(q)-f(k)}{q-k} + \mathrm{PV}\int_0^\Lambda \mathrm{d}q\,\frac{f(k)}{q-k} \tag{19}$$

  - in the first term, the numerator cancels the singularity
  - the integral in the second term can be solved analytically

Numerical principal value integration

- overall, we have arrived at

$$\mathrm{PV}\int_0^\Lambda \mathrm{d}q\, \underbrace{\frac{f(q)-f(k)}{q-k}}_{\tilde{f}\,(q)} + \mathrm{PV}\int_0^\infty \mathrm{d}q\, \frac{f(k)}{q-k} + \mathrm{i}\pi f(k) \tag{20}$$

  - to the first term, we can now apply a standard quadrature: $\to \sum w_i \tilde{f}\,(q_i)$
  - after this step, we can split up $\tilde{f}$ again because it's all finite sums!
  - the second term is simply $f(k)\log\left(\frac{\Lambda-k}{k}\right)$

- eventually, we arrive at the original integral (first part of $\tilde{f}$), and a sum of three terms multiplying $f(k)$:

$$R = \sum_i \frac{w_i}{q_i - k} + \log\left(\frac{\Lambda-k}{k}\right) + \mathrm{i}\pi \tag{21}$$

- this can be interpreted as adding an additional point $q_0 = k$ with weight $w_0 = R$ to the original quadrature mesh $\{q_i\}_{i=1}^N$

Partial-wave projection

- for V spherically symmetric, it makes sense to expand the T-matrix in partial waves:

$$T(E_{\mathbf{k}};\mathbf{p},\mathbf{k}) = \sum_{l=0}^{\infty}(2l+1)T_l(E_k;p,k)P_l(\cos\theta) \tag{22}$$

- $\cos\theta$ here denotes the angle between $\mathbf{k}$ and $\mathbf{p}$
- the l-th partial-wave projection is then given by

$$T_l(E_k;p,k) = \frac{1}{2}\int \mathrm{d}\cos\theta\, P_l(\cos\theta)T(E_{\mathbf{k}};\mathbf{p},\mathbf{k}) \tag{23}$$

- this projection can be applied to the Lippmann-Schwinger equation as a whole:

$$\begin{aligned} T_l(E_k+\mathrm{i}\varepsilon;p,k) = V_l(p,k) \\ + \int \frac{dq\,q^2}{2\pi^2} V_l(p,q)G_0(E_k+\mathrm{i}\varepsilon;q)T_l(E_k+\mathrm{i}\varepsilon;q,k) \end{aligned} \tag{24}$$

- this involves the partial-wave projected potential $V_l(p,k)$

Scattering amplitude

- the **scattering amplitude** is now defined, for $|\mathbf{k}'| = |\mathbf{k}| = k$, as

$$f(\mathbf{k}', \mathbf{k}) = f_k(\theta) = -\frac{\mu}{2\pi} \langle \mathbf{k}' | V | \psi_{\mathbf{k}}^{(+)} \rangle = -\frac{\mu}{2\pi} \langle \mathbf{k}' | T | \mathbf{k} \rangle \tag{25}$$

- physically, it describes the modulation of the outgoing scattered wave
- it has an expansion into partial waves analogous to the T-matrix
- the on-shell T-matrix is related to the partial-wave scattering amplitude, and also to the partial-wave S-matrix

$$f_l(k) = -\frac{\mu}{2\pi} T_l(E_k; k, k) = \frac{S_l(k) - 1}{2\mathrm{i}k} \tag{26}$$

Notes

- conventions regarding the prefactors in Eq. (26) may differ in the literature
- in the three-dimensional vector representation, the on-shell point is defined as $T(E_{\mathbf{k}}; \mathbf{k}', \mathbf{k})$ with $|\mathbf{k}'| = |\mathbf{k}|$, leaving a dependence on θ

Scattering phase shift

- unitarity of the partial-wave S-matrix, $|S_l(k)| = 1$, implies that

$$S_l(k) = \mathrm{e}^{2\mathrm{i}\delta_l(k)} \tag{27}$$

- this defines the **scattering phase shift** $\delta_l(k)$
- the factor of two in the exponent is a (convenient) convention at this point
- from this form we obtain further useful ways to express the scattering amplitude:

$$f_l(k) = \frac{\mathrm{e}^{2\mathrm{i}\delta_l(k)} - 1}{2\mathrm{i}k} = \frac{\mathrm{e}^{\mathrm{i}\delta_l(k)} \sin\delta_l(k)}{k} = \frac{1}{k\cot\delta_l(k) - \mathrm{i}k} \tag{28}$$

- the final form in Eq. (28) is particularly useful to calculate the phase shift from the T-matrix:

$$\delta_l(k) = \operatorname{arccot}\left(-\frac{2\pi}{\mu k} T(E_k; k, k)^{-1} + \mathrm{i}\right) \tag{29}$$

- the $\mathrm{i}k$ in the Eq. (28) is directly related to the $\mathrm{i}\pi$ in the principal-value formula

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Solution of Lippmann-Schwinger equation

Effective range expansion

- the scattering amplitude can be expanded in p, and it can be shown that for the l-th partial wave, $f_l(k) = \mathcal{O}(k^{2l})$, so:

$$f_l(p) = \frac{1}{k \cot \delta_l(k) - \mathrm{i}k} = \sum_{n=2l}^{\infty} c_{l,n} k^n \tag{30}$$

- this implies a **hierarchy of partial waves**
  - larger l become subsequently important as k increases
  - only S-waves ($l = 0$) contribute for $k \to 0$
- odd powers in Eq. (30) actually only arise from the unitarity cut, $-\mathrm{i}k$
- all nontrivial physics information is contained in $k \cot \delta_l(k)$, and this quantity is analytic in $k^2 \sim E$
- conventionally one defines the **effective range expansion** as

$$k^{2l+1} \cot \delta_l(k) = -\frac{1}{a_l} + \frac{r_l}{2} k^2 + \mathcal{O}(k^4) \tag{31}$$

- from this one can infer that $c_{l,0} = a_l$

Scattering wavefunctions

- from the half off-shell T-matrix, we can also obtain **scattering wavefunctions** in momentum space
- recall the initial form of the Lippmann-Schwinger equation, Eq. (8):

$$|\psi_{\mathbf{k}}^{(+)}\rangle = |\mathbf{k}\rangle + (E_{\mathbf{k}} - H_0 + \mathrm{i}\varepsilon)^{-1} V |\psi_{\mathbf{k}}^{(+)}\rangle$$

- with $V|\psi_{\mathbf{k}}^{(+)}\rangle = T|\mathbf{k}\rangle$, we obtain directly:

$$\langle \mathbf{q}|\psi_{\mathbf{k}}^{(+)}\rangle = \langle \mathbf{q}|\mathbf{k}\rangle + \langle \mathbf{q}|(E_{\mathbf{k}} - H_0 + \mathrm{i}\varepsilon)^{-1} T|\mathbf{k}\rangle \tag{32}$$

- from the first term $\langle \mathbf{q}|\mathbf{k}\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{q} - \mathbf{k})$ it is clear that this is a distribution, not an ordinary function
- the second term, with $\varepsilon \to 0$ implied, contains a smooth part as well as a pole contribution (from the on-shell point):

$$\langle \mathbf{q}|(E_{\mathbf{k}} - H_0 + \mathrm{i}\varepsilon)^{-1} T|\mathbf{k}\rangle = \frac{2\mu T(E_{\mathbf{k}}; \mathbf{q}, \mathbf{k})}{\mathbf{k}^2 - \mathbf{q}^2 + \mathrm{i}\varepsilon} \tag{33}$$

Scattering wavefunctions

- altogether we have

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{q}) = \langle \mathbf{q} | \psi_{\mathbf{k}}^{(+)} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{q} - \mathbf{k}) + \frac{2\mu T(E_{\mathbf{k}}; \mathbf{q}, \mathbf{k})}{\mathbf{k}^2 - \mathbf{q}^2 + \mathrm{i}\varepsilon} \tag{34}$$

- Fourier-transform yields **wavefunctions in configuration space**:

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{r}) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + 2\mu \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}} \frac{T(E_{\mathbf{k}}; \mathbf{q}, \mathbf{k})}{\mathbf{k}^2 - \mathbf{q}^2 + \mathrm{i}\varepsilon} \tag{35}$$

Notes

- this $\psi_{\mathbf{k}}^{(+)}(\mathbf{r})$ is an ordinary function, but it is still not normalizable and therefore not an element of the Hilbert space $L^2(\mathbb{R}^3)$
- the integral in Eq. (35) goes across the pole at $\mathbf{q} = \mathbf{k}$, so it is again defined as a principal value
- numerically this is handled by exactly the same modified quadrature rule as before

Scattering eigenstates

- the Schrödinger equation asserts that

$$(H_0 + V)|\psi_{\mathbf{k}}^{(+)}\rangle = E_p|\psi_{\mathbf{k}}^{(+)}\rangle \tag{36}$$

- in configuration space, where $\psi_{\mathbf{k}}^{(+)}(\mathbf{r})$ is an ordinary function and H is a differential operator, we can test this by looking at the ratio $(H\psi_{\mathbf{k}}^{(+)})(\mathbf{r})/\psi_{\mathbf{k}}^{(+)}(\mathbf{r})$
- in momentum space, the analog of this is not well defined because we are dealing with distributions:

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{q}) = (2\pi)^3\delta^{(3)}(\mathbf{q} - \mathbf{k}) + \cdots \tag{37}$$

- nor can we take matrix elements of $|\psi_{\mathbf{k}}^{(+)}\rangle$
- the resolution is that scattering states are really **generalized eigenstates** of the Hamiltonian, in the sense that for any test function $\phi(\mathbf{q}) = \langle\phi|\mathbf{q}\rangle$ it holds that

$$\langle\phi|(H_0 + V)|\psi_{\mathbf{k}}^{(+)}\rangle = E_k\langle\phi|\psi_{\mathbf{k}}^{(+)}\rangle \tag{38}$$

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Eigenstate verification

Bound states

- we will formally derive later that bound states correspond to **poles of the S-matrix** at negative energies
- since, schematically, $S = \mathbf{1} + T$, these poles really need to be in the T-matrix
- moreover, it can be shown that these poles are necessarily simple poles
- therefore, assuming the existence of a bound state at energy $E = -B$, we can parametrize the T-matrix in the vicinity of the pole as follows:

$$T(E; p, p') \sim \frac{Z(p, p')}{E + B} \quad \text{for} \quad E \to -B \tag{39}$$

- finally, it can be shown that the residue $Z(p, p')$ factorizes: $Z(p, p') = B^*(p)B(p')$
- we call $B(p)$ the vertex fuction associated with the bound state, and we will later derive its relation to the momentum-space wavefunction of the bound state
- in abstract operator notation, we write Eq. (39) as

$$T(E) \sim \frac{|B\rangle\langle B|}{E + B} \quad \text{for} \quad E \to -B \tag{40}$$

Bound-state equation

- inserting the factorized pole form into the Lippmann-Schwinger equation gives:

$$\frac{B^*(p)B(p')}{E+B} = V(p,p') + \int \frac{dq\, q^2}{2\pi^2} V(p,q) G_0(E;q) \frac{B^*(q)B(p')}{E+B} \tag{41}$$

- we have used here the partial-wave projected form and dropped the subscript l
- next, we multiply through with $(E+B)$ and take the limit $E \to -B$
  - the potential does not have any poles, so $\lim_{E\to -B} V(p,p') = 0$
  - the free Green's function is also regular for $E < 0$

$$B^*(p)B(p') = \int \frac{\mathrm{d}q\, q^2}{2\pi^2} V(p,q) G_0(-B;q) B^*(q)B(p') \tag{42}$$

- finally, we can drop the common factor $B(p')$ on both sides and use that both V and G_0 are real to obtain the **Schrödinger equation for the vertex function**:

$$B(p) = \int \frac{\mathrm{d}q\, q^2}{2\pi^2} V(p,q) G_0(-B;q) B(q) \tag{43}$$