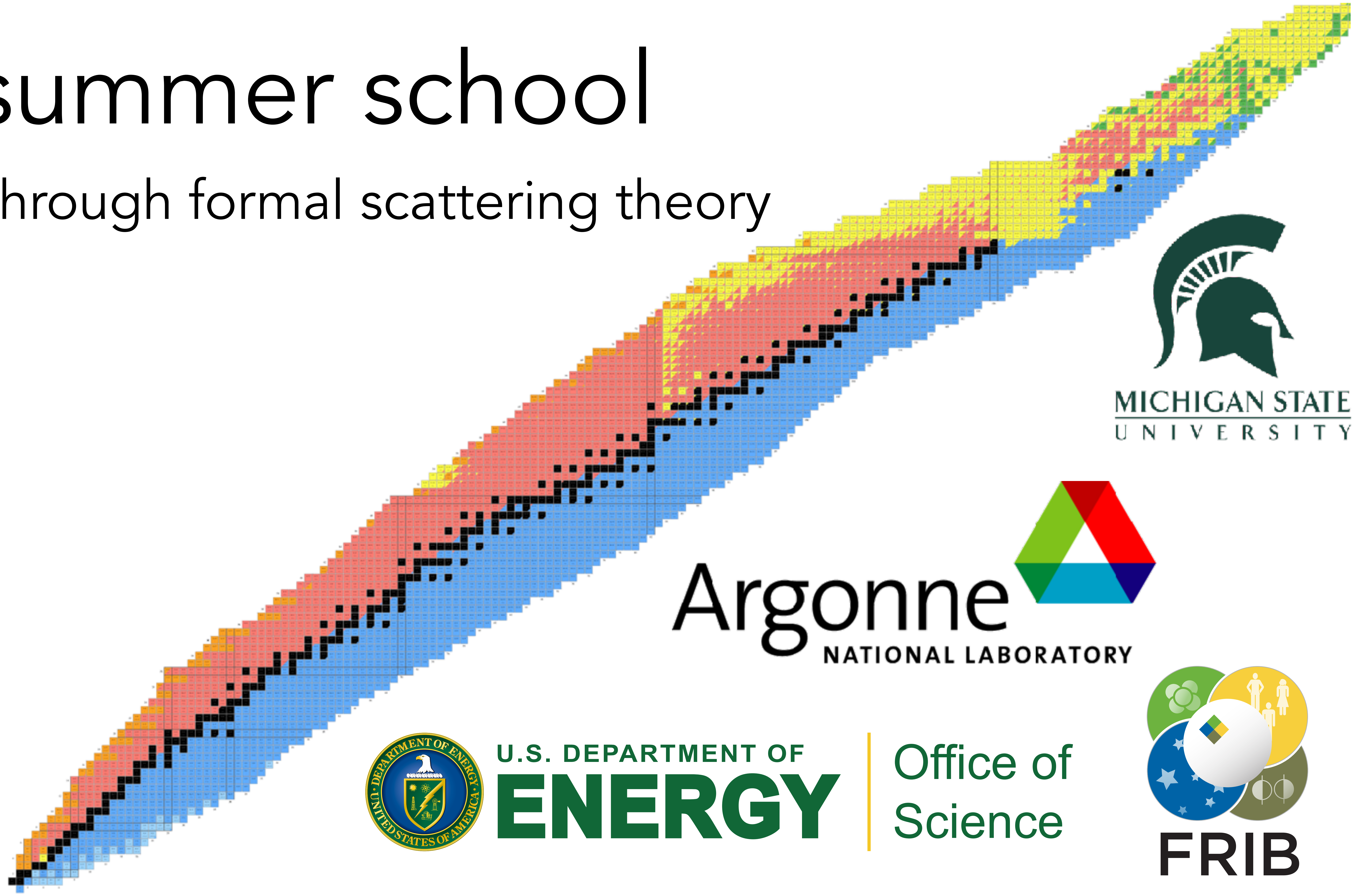
FRIB-TA summer school

A practical walk through formal scattering theory



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Newton basis (D3-M1, 30 min)

An interesting consequence of the RHS is that it justifies a completeness relation involving Gamow states called the Berggren basis.

First, we remind that it was shown with some effort that it is possible to form a complete basis using bound and scattering states called the **Newton basis:**

$$\sum_{i=(b)} |u(k_i)\rangle\langle u(k_i)| + \int_0^\infty dk\, |u(k)\rangle\langle u(k)| = \hat{1}$$

where the sum goes over all the bound states $k_i = +i|k_i|$ of a given potential, and the integral goes over all its scattering states $k = |k|$.

Already this formula makes use of scattering states which are not in the Hilbert space, but at least all the states have real energies.

Berggren basis

However, if one tries to expand a resonance wave function using only real energy wave functions, it is clear that it will be impossible to have the imaginary part necessary to describe decay in the quasi-stationary formalism.

Can we extend the Newton basis by analytical continuation as to include explicitly Gamow states? This question led T. Berggren to formulate the following basis:

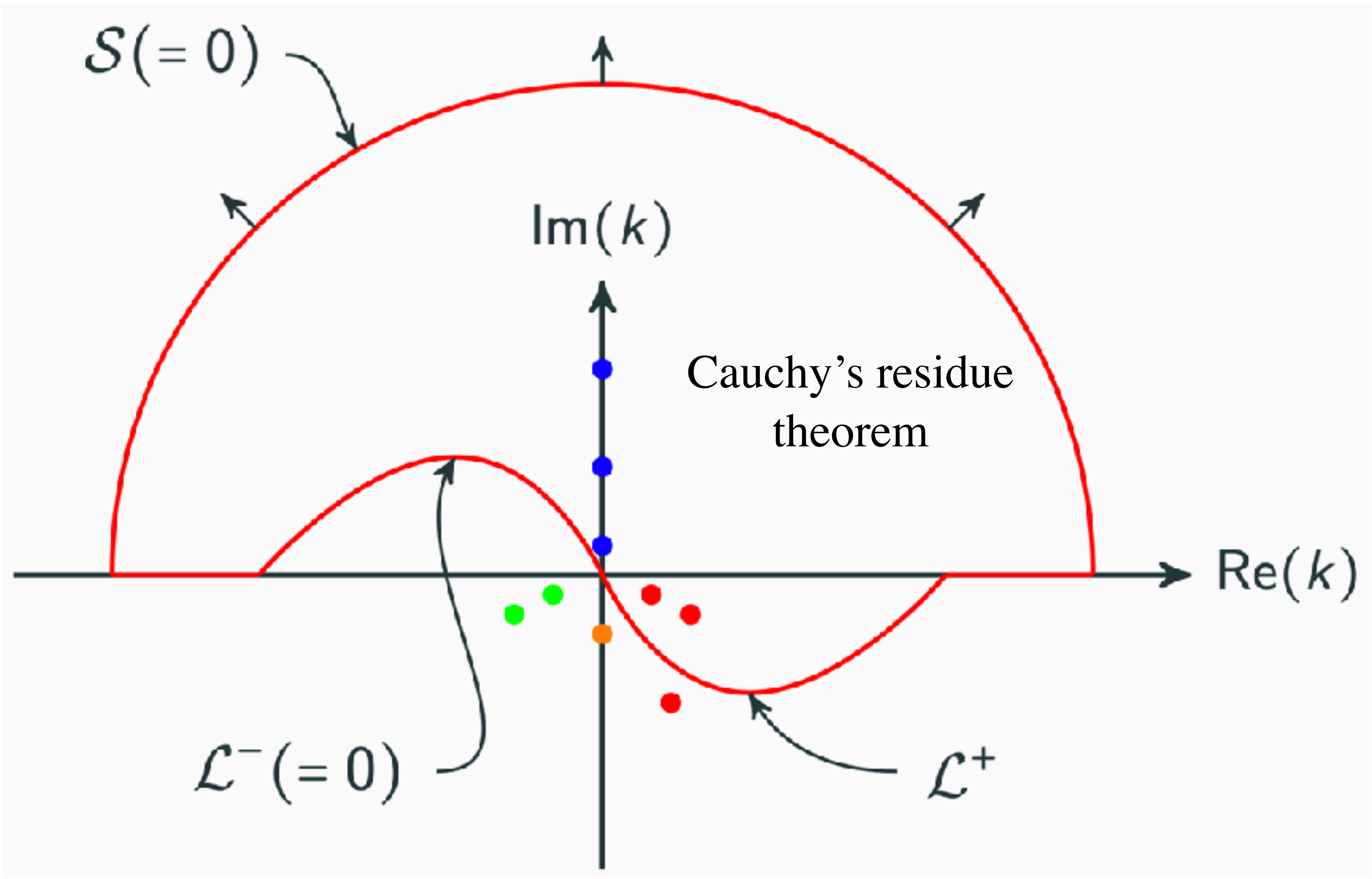
$$\sum_{i=(b,r)} |u(k_i)\rangle\langle\tilde{u}(k_i)| + \int_{L^+} dk\, |u(k)\rangle\langle\tilde{u}(k)| = \hat{1}$$

where the sum goes over all the bound states $k_i = +i|k_i|$ and **selected resonances**, and the integral goes along a path L^+ in the lower-half of the complex momentum plane starting at $k = 0$ and surrounding the selected resonances before extending to infinity.

Berggren basis

Here is the schematic method used by T. Berggren:

T. Berggren, Nucl. Phys. A **109** 265 (1968)



A much simpler way to obtain the same result is to simply note that:

$$\int_{L^+} dk \,|\, u(k)\rangle\langle\tilde{u}(k)\,| - \int_0^\infty dk \,|\, u(k)\rangle\langle u(k)\,|$$
$$= 2\pi i \sum_r \mathbf{Res}_{k=k_r}[\,|\, u(k)\rangle\langle\tilde{u}(k)\,|\,]$$

Then, one needs to calculate the residues.

Berggren basis

Using the normalization of scattering states and the S-matrix equality:

$$C_l^+(k)C_l^-(k) = \frac{1}{2\pi} \qquad\qquad S_l(k) = -\frac{C_l^+(k)}{C_l^-(k)} = \frac{J_l^-(k)}{J_l^+(k)}$$

It follows that:

$$C_l^{\pm}(k) = \sqrt{-\frac{1}{2\pi}\frac{J_l^{\mp}(k)}{J_l^{\pm}(k)}}$$

The asymptotic wave function is thus:

$$u_l(k,r) = \sqrt{-\frac{1}{2\pi}\frac{J_l^-(k)}{J_l^+(k)}}u_l^+(k,r) + \sqrt{-\frac{1}{2\pi}\frac{J_l^-(k)}{J_l^+(k)}}u_l^-(k,r)$$

Berggren basis

When approaching a pole $k \to k_p$, one has $J_l^+(k) \to 0$ and thus:

$$|u(k)\rangle\langle\tilde{u}(k)| \sim -\frac{1}{2\pi}\frac{J_l^-(k)}{J_l^+(k)}|u^+(k)\rangle\langle\tilde{u}^+(k)|$$

Using the Taylor expansion of the Jost function and the formula for its derivative:

$$J_l^+(k) \approx (k-k_p)\frac{dJ_l^+(k)}{dk}\bigg|_{k=k_n} \qquad \frac{dJ_l^+(k)}{dk}\bigg|_{k=k_n} = iJ_l^-(k_p)\int_0^\infty dr\,[u_l^+(k_p,r)]^2$$

one obtains:

$$|u(k)\rangle\langle\tilde{u}(k)| \sim -\frac{1}{2\pi}\frac{J_l^-(k_p)}{(k-k_p)}\frac{|u^+(k_p)\rangle\langle\tilde{u}^+(k_p)|}{iJ_l^-(k_p)\int_0^\infty dr\,[u_l^+(k_p,r)]^2}$$

Berggren basis

Since by definition a normalized pole state is just:

$$|u(k_p)\rangle\langle\tilde{u}(k_p)| = \frac{|u^+(k_p)\rangle\langle\tilde{u}^+(k_p)|}{\int_0^\infty dr\,[u_l^+(k_p,r)]^2}$$

we have:

$$|u(k)\rangle\langle\tilde{u}(k)| \sim -\frac{1}{2\pi i}\frac{1}{k-k_p}|u(k_p)\rangle\langle\tilde{u}(k_p)|$$

Clearly:

$$\lim_{k\to k_p}(k-k_p)|u(k)\rangle\langle\tilde{u}(k)| \sim -\frac{1}{2\pi i}|u(k_p)\rangle\langle\tilde{u}(k_p)| = \mathop{\mathbf{Res}}_{k=k_p}[\,|u(k)\rangle\langle\tilde{u}(k)|\,]$$

Berggren basis

Using the residue obtained, the Berggren basis follows immediately since the contour gives:

$$\begin{aligned}\int_0^\infty dk\,|u(k)\rangle\langle u(k)| &= \int_{L^+} dk\,|u(k)\rangle\langle\tilde{u}(k)| - 2\pi i \sum_r \underset{k=k_r}{\mathbf{Res}}[\,|u(k)\rangle\langle\tilde{u}(k)|\,] \\ &= \int_{L^+} dk\,|u(k)\rangle\langle\tilde{u}(k)| + \sum_r |u(k_r)\rangle\langle\tilde{u}(k_r)|\end{aligned}$$

Then we just add the sum over the bound states, and we finally have **a single particle basis including bound states, resonances, and scattering states.**

Berggren basis

Here, we will use the Berggren basis as an example of method to describe many-body resonances, but there are other ways to do just that.

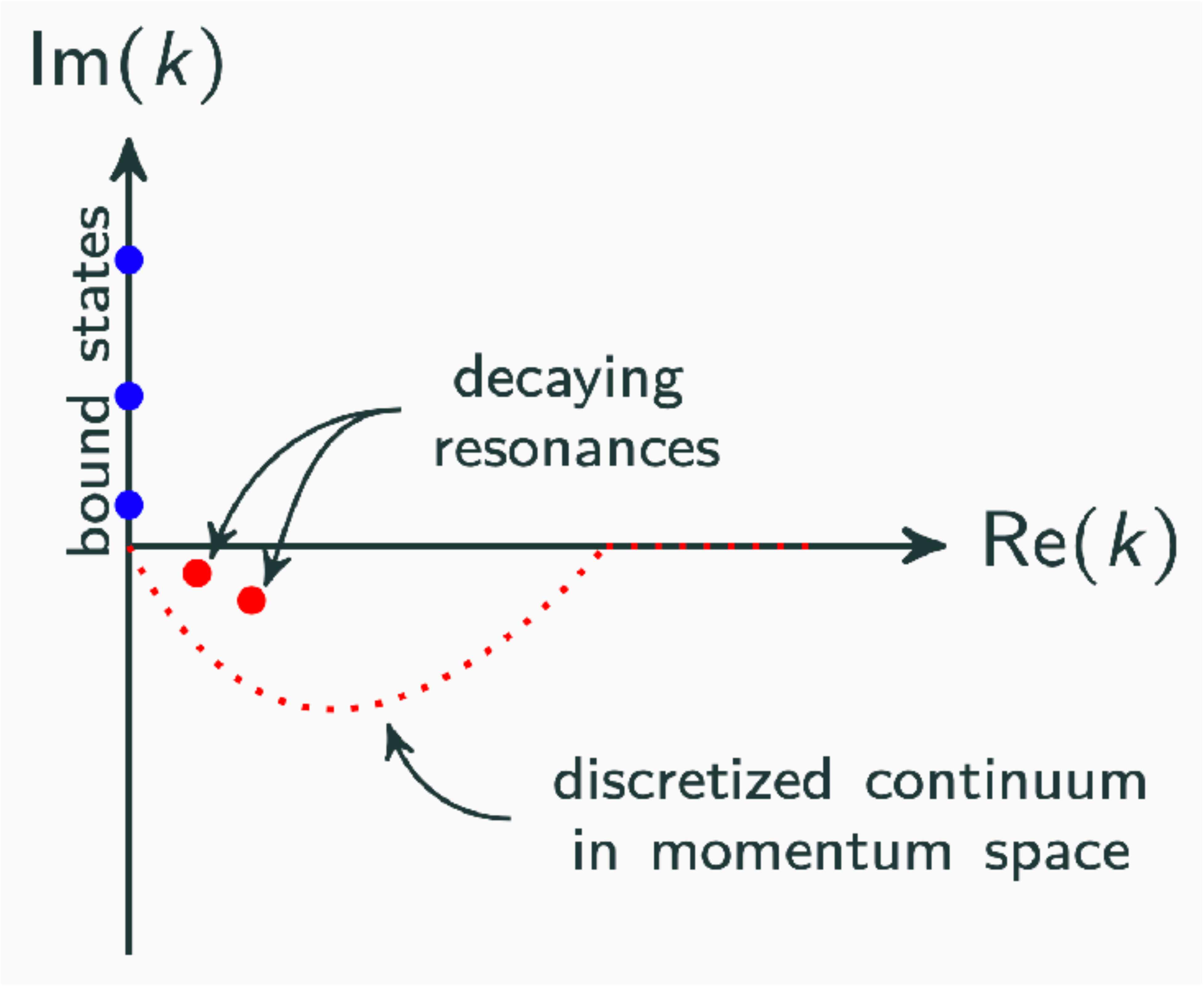
For instance, one could solve the many-body Schrödinger equation in momentum space, use the uniform complex-scaling method, or calculate phase-shifts from the resonating group method to extract resonance information.

In practical applications, the contour L^+ of scattering states can be discretized by using a quadrature method for the integral:

$$\int_{L^+} dk \, | u(k) \rangle \langle \tilde{u}(k) | + \sum_r | u(k_r) \rangle \langle \tilde{u}(k_r) | \approx \sum_i | u(k_i) \rangle \langle \tilde{u}(k_i) |$$

Berggren basis

In practical applications, the contour L^+ of scattering states can be discretized by using a quadrature method for the integral:



$$\int_{L^+} dk\, |u(k)\rangle\langle\tilde{u}(k)| + \sum_r |u(k_r)\rangle\langle\tilde{u}(k_r)| \approx \sum_i |u(k_i)\rangle\langle\tilde{u}(k_i)|$$

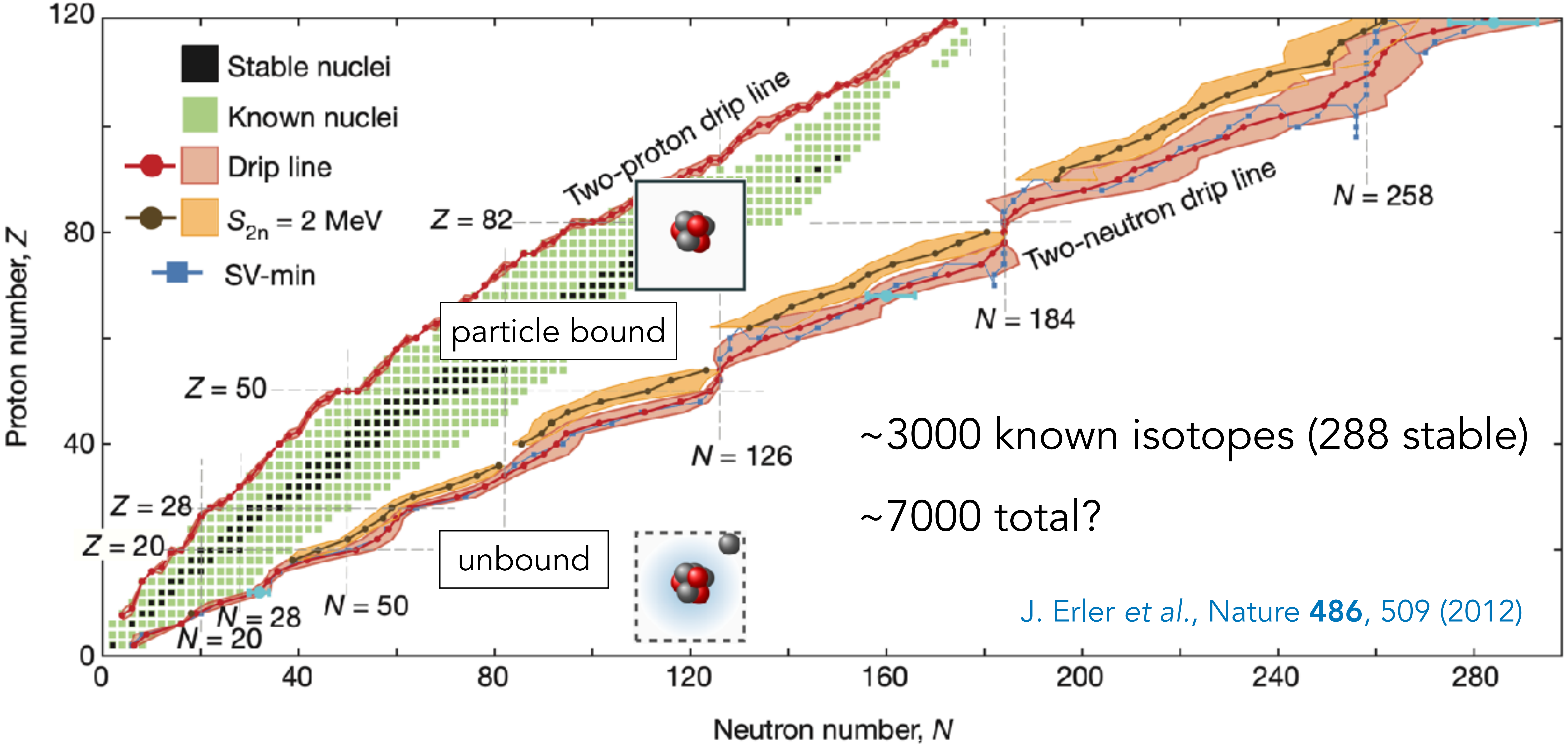
One can then perform **many-body calculations in the quasi-stationary framework.**

The resulting physical many-body energies are complex and satisfy:

$$E = E_r - i\frac{\Gamma}{2}$$

Low-energy nuclear theory: challenges

Limits of nuclear stability: How many nuclei can exist?



Physics of exotic nuclei.

C. Johnson and K. Launey (eds.), J. Phys. G **47** 123001 (2020)

Emergent properties:

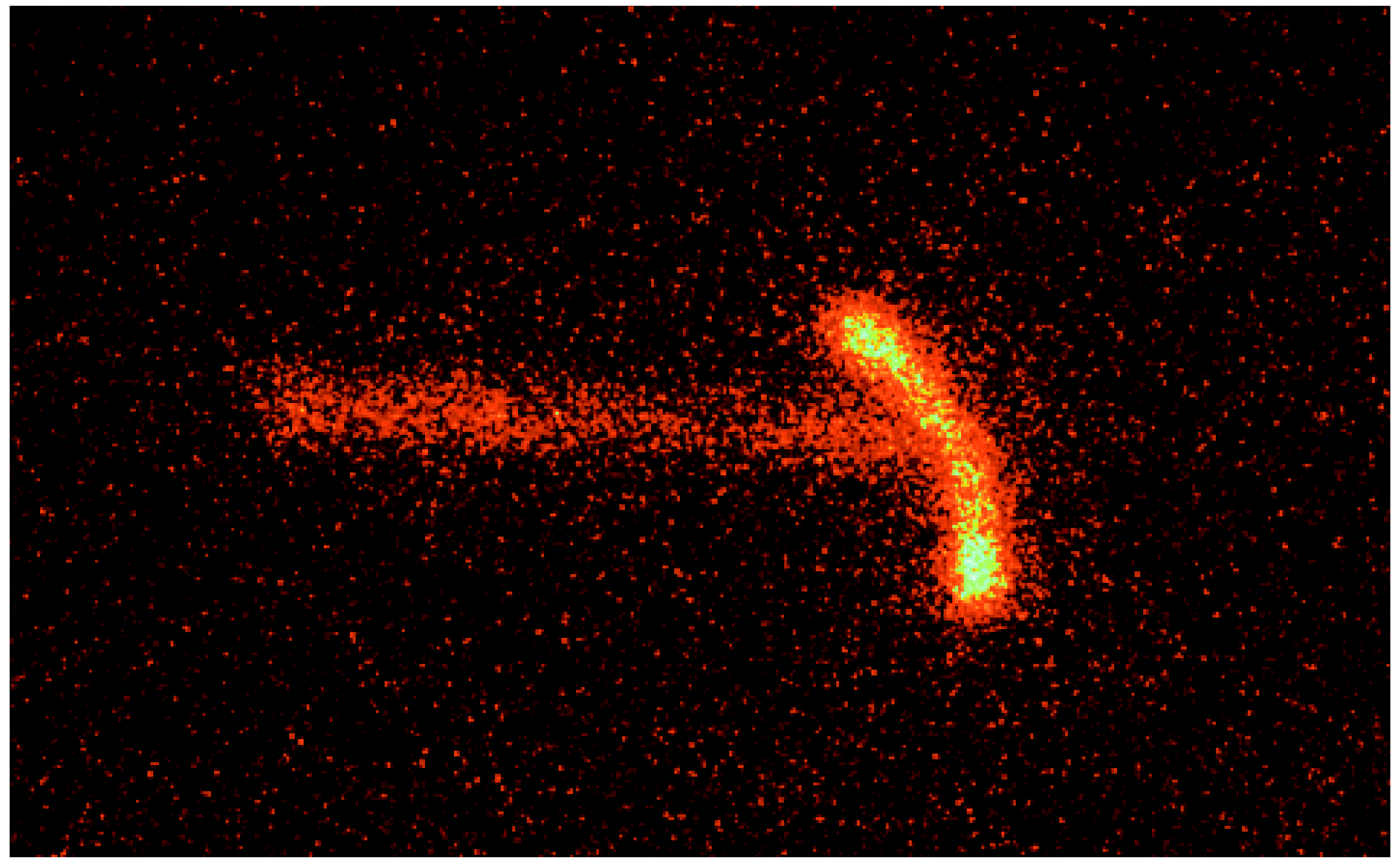
What is their origin?

- Shell structure
- Deformation
- Rotational motion
- Clustering

- Halos
- Exotic decay modes
- Resonances

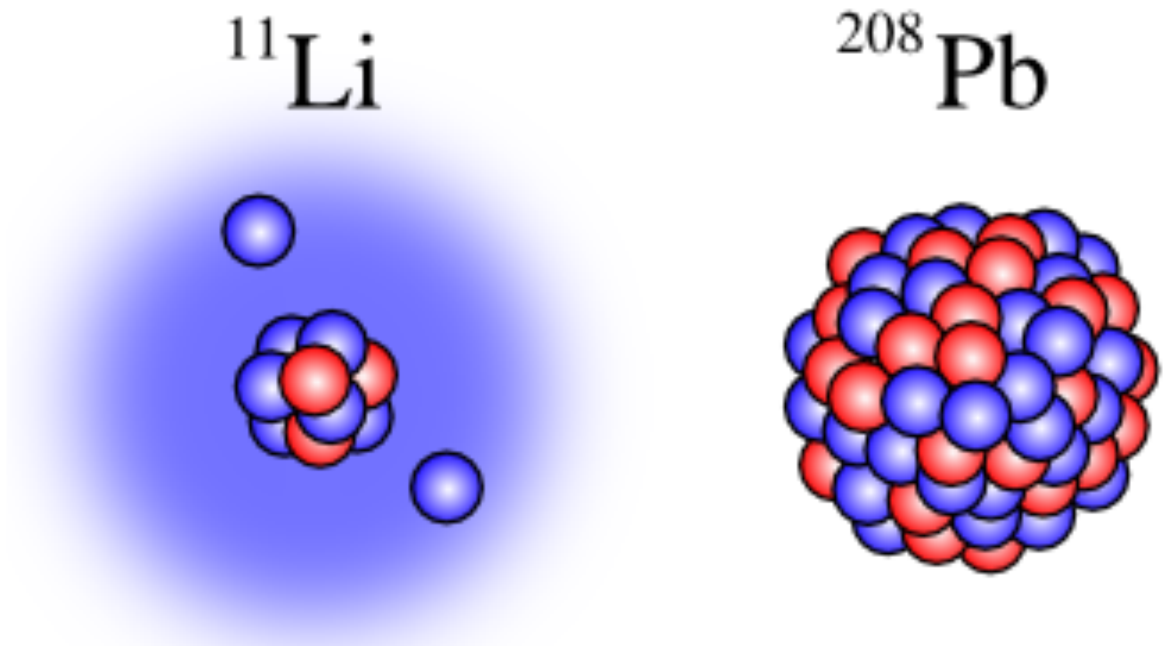
Low-energy nuclear theory: challenges

Exotic decay modes.



K. Miernik *et al.*, Phys. Rev. Lett. **99**, 192501 (2007).

Halo structures



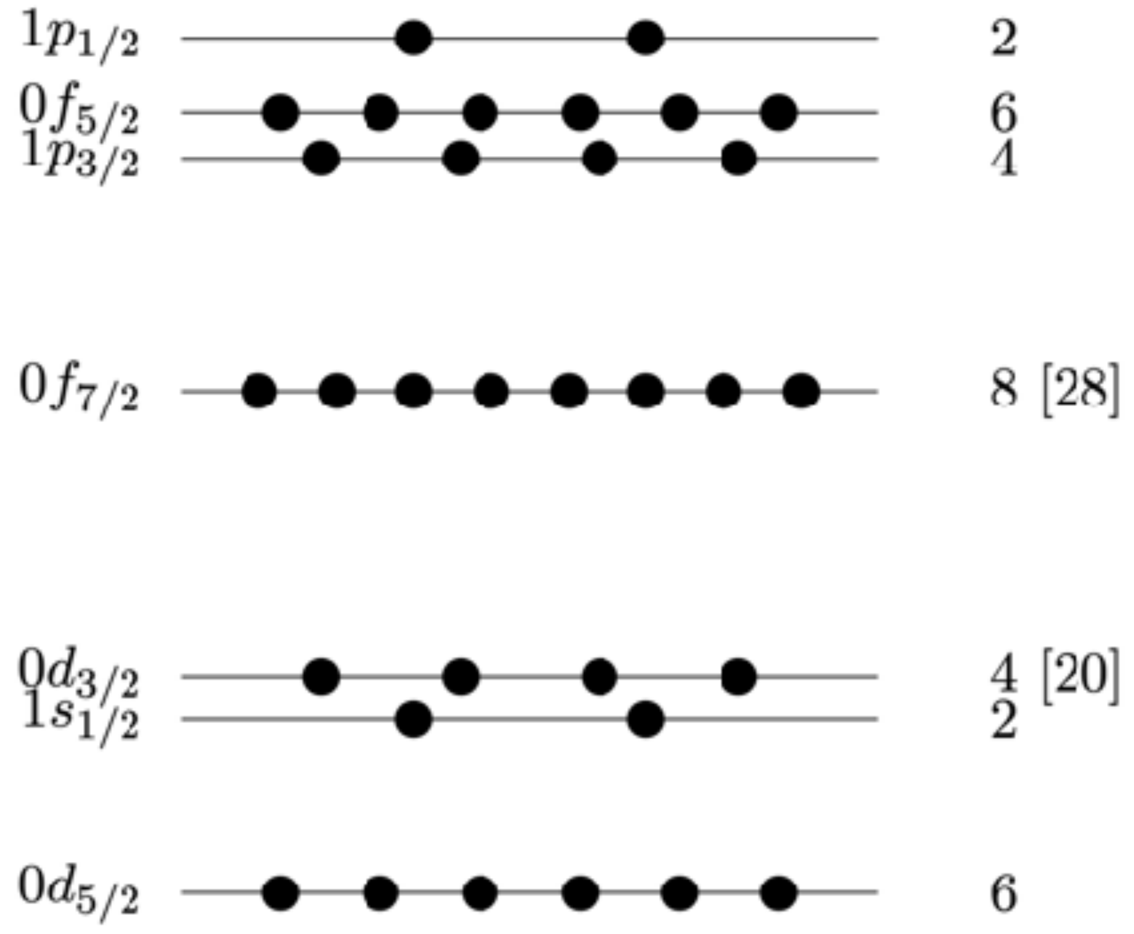
I. Tanihata *et al.*, Phys. Rev. Lett **55**, 2676 (1985)

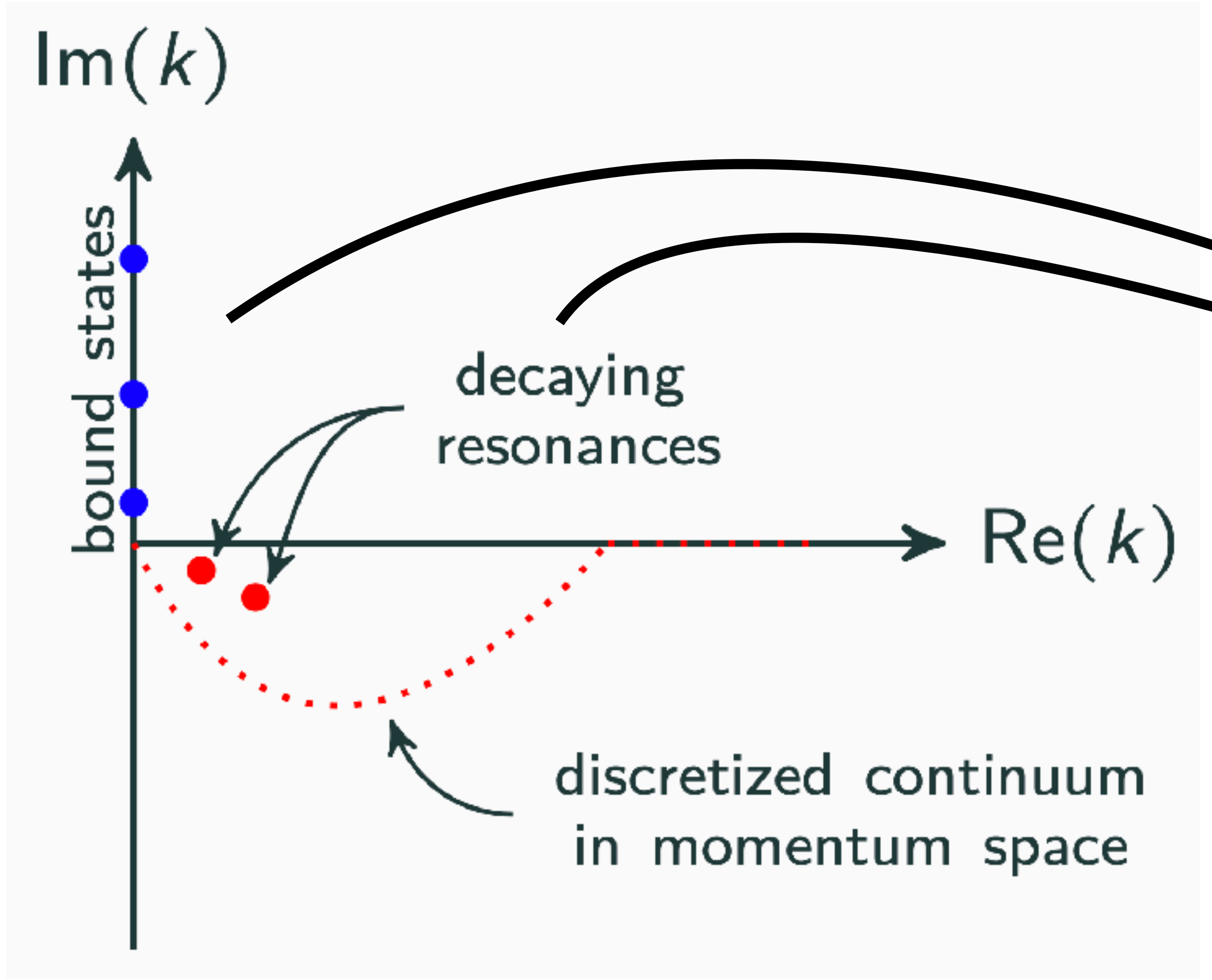
Gamow shell model

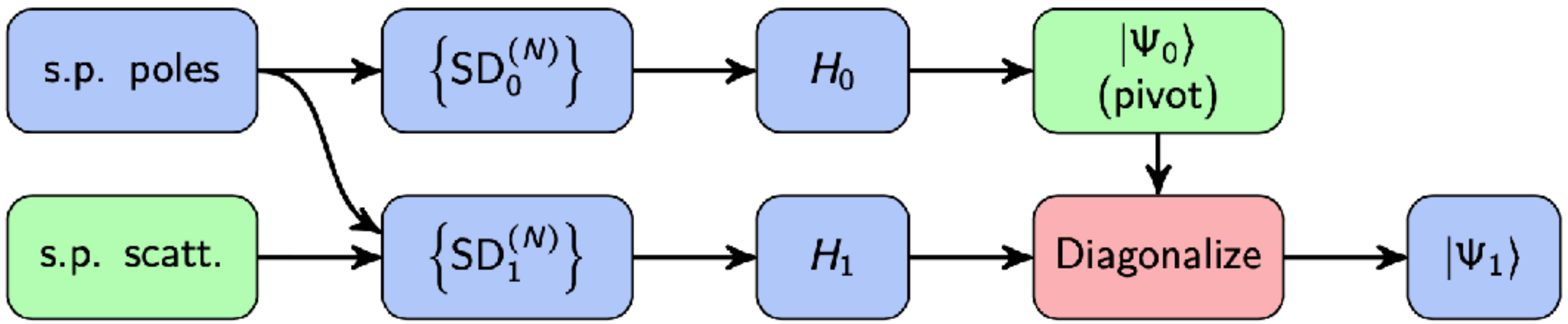
First many-body method using the Berggren basis.

N. Michel *et al.*, J. Phys, G **36**, 013101 (2009)

- Configuration interaction method: $|\Psi\rangle = \sum_i c_i |\mathbf{SD}_i\rangle$
- Complex-symmetric Hamiltonian matrix. $\hat{H}|\Psi\rangle = \tilde{E}|\Psi\rangle$







Other related approaches

The Berggren basis was also used in:

- The density matrix renormalization group method.
- The coupled-cluster and in-medium similarity renormalization group methods.
- The Jacobi coupled-channel three-body method and particle-plus-rotor model.

In all these methods, the same difficulties are present:

- Identification of physical states, i.e. poles of the many-body S-matrix.
- Reconstruction of the many-body asymptotic (width) from the single-particle states.

This is why it is important to study resonances using other methods as well (reactions, k-space…).

Nuclear forces

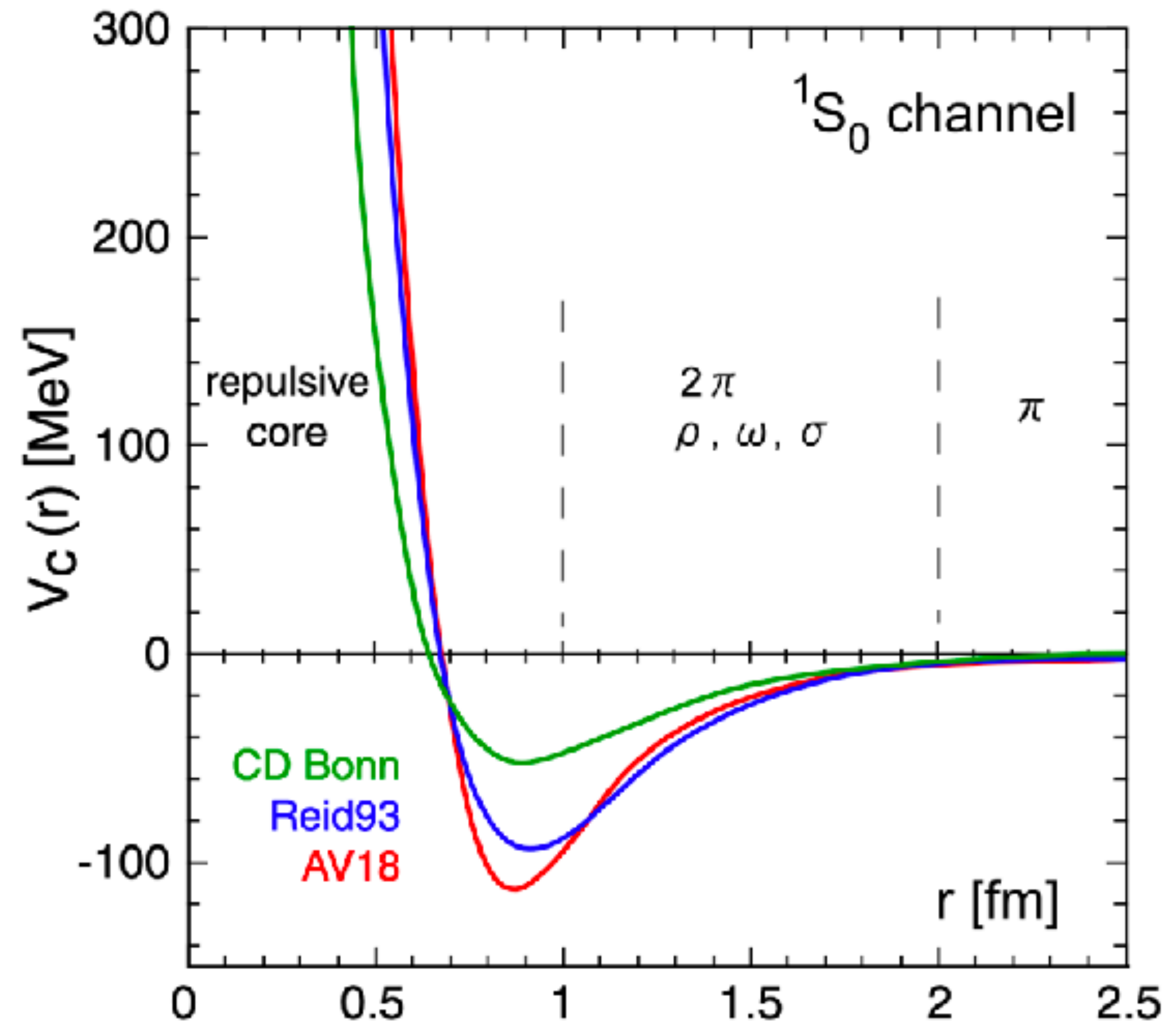
Fixing models of nuclear forces is another important use of scattering theory in modern research.

$$\hat{H} = \sum_{ij} \langle i|T|j\rangle \hat{a}_i^\dagger \hat{a}_j + \frac{1}{(2!)^2} \sum_{ijkl} \langle ij|V_{\mathrm{NN}}^{\mathrm{as}}|kl\rangle \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_l \hat{a}_k + \frac{1}{(3!)^2} \sum_{ijklmn} \langle ijk|V_{\mathrm{NNN}}^{\mathrm{as}}|lmn\rangle \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k^\dagger \hat{a}_n \hat{a}_m \hat{a}_l + \ldots$$

kinetic energy | 2-body | 3-body

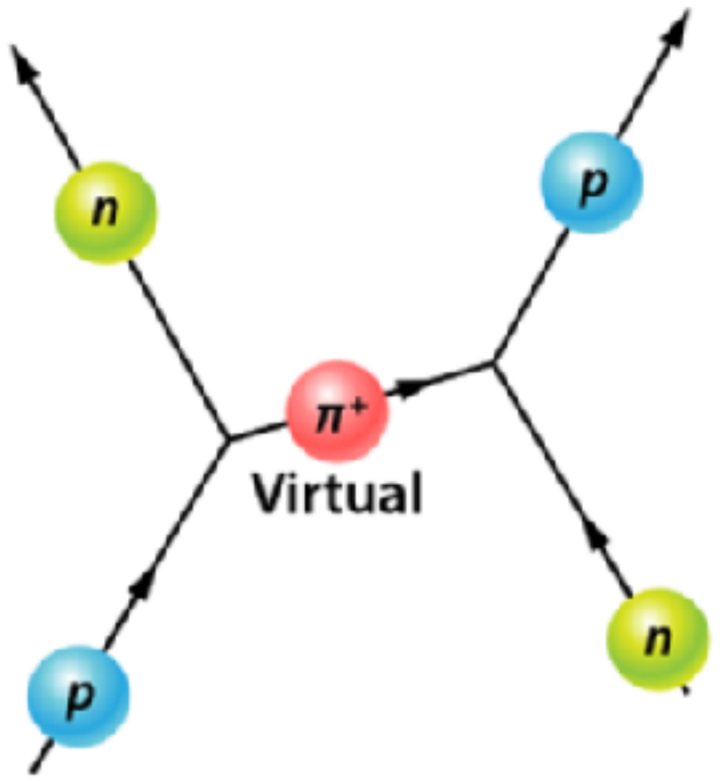
The specific form of nuclear forces is unknown as it would require to solve the non-perturbative quantum chromodynamics equation at low energy, but this is not yet feasible.

Instead, a potential between two (or more) nucleons is used to model their interaction: **this is a scattering problem.**



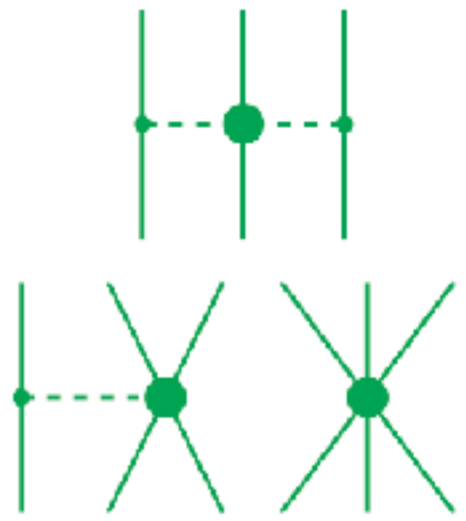
K. Fossez, FRIB & ANL

Nuclear forces



Nuclear potentials can be guessed or derived from a set of principles.

R. Machleidt and D. R. Entem, Phys. Rep. **503**, 1 (2011)

	NN	3N
LO $(Q/\Lambda_\chi)^0$		
NLO $(Q/\Lambda_\chi)^2$		
NNLO $(Q/\Lambda_\chi)^3$		

Then, using a partial wave expansion, the phase shifts associated with the scattering of two interacting nucleons can be expressed and adjusted to experiment.

This process guarantees that the resulting potential will have the desired properties up to some energy.

Nuclear forces