A practical walk through formal scattering theory

Connecting bound states, resonances, and scattering states in exotic nuclei and beyond

Contour rotation

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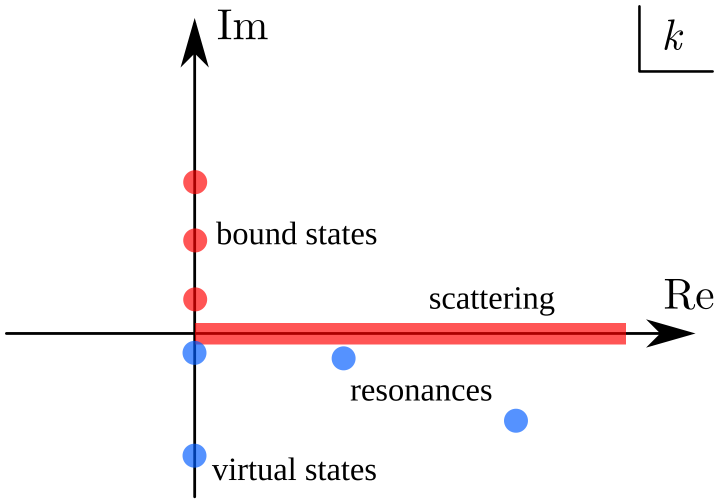
Virtual states

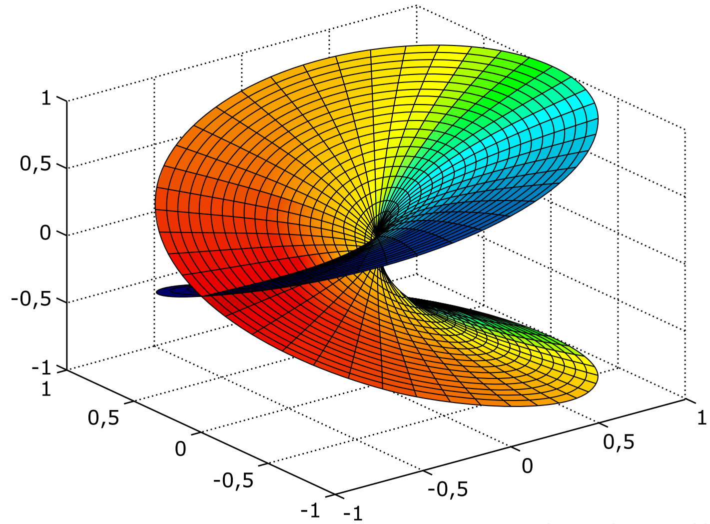
- as mentioned before, complex momenta $k = -\mathrm{i}\kappa$ also yield negative energies
- S-matrix poles at such positions in the complex k plane are called **virtual states** (or antibound states)



- as a function of energy, the S-matrix has multiple branches: $S_l^{\mathrm{I}}(E)$, $s_l^{\mathrm{II}}(E)$
  - bound states are poles of $S_l^{\mathrm{I}}(E)$ for negative E, $k = \mathrm{i}\kappa$
  - virtual (antibound) states are poles of $S_l^{\mathrm{II}}(E)$ for negative E, $k = -\mathrm{i}\kappa$
  - other poles of $S_l^{\mathrm{II}}(E)$ are resonances

Riemann sheets

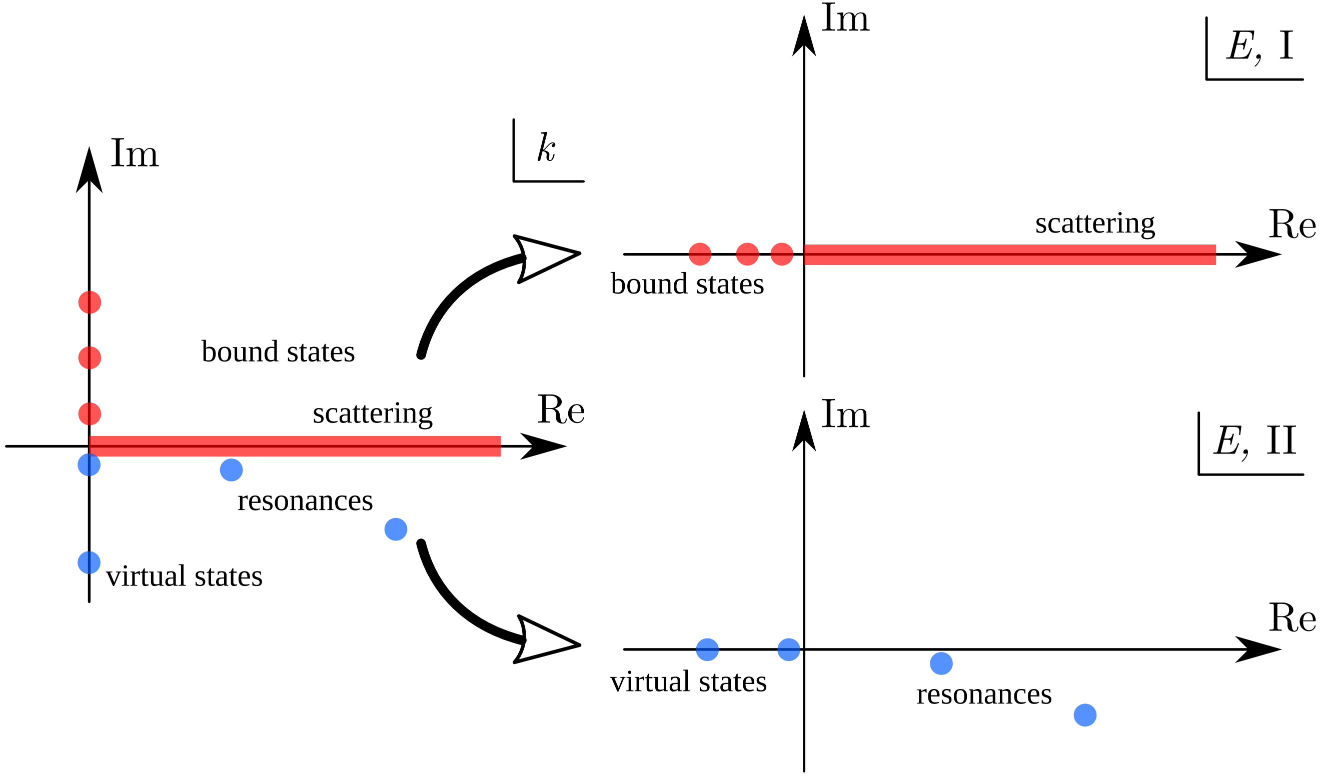
- recall that for $E = k^2$, $\sqrt{E}$ can equally well be defined as $+k$ or $-k$
- these are the two branches of the square root function
- typically, the principal branch is taken to be the positive solution
- both branches can be combined by defining $\sqrt{E}$ on a **Riemann surface**
  - in this case, it is built out of two Riemann sheets
  - these are connected at the branch cut, chosen along the negative real axis



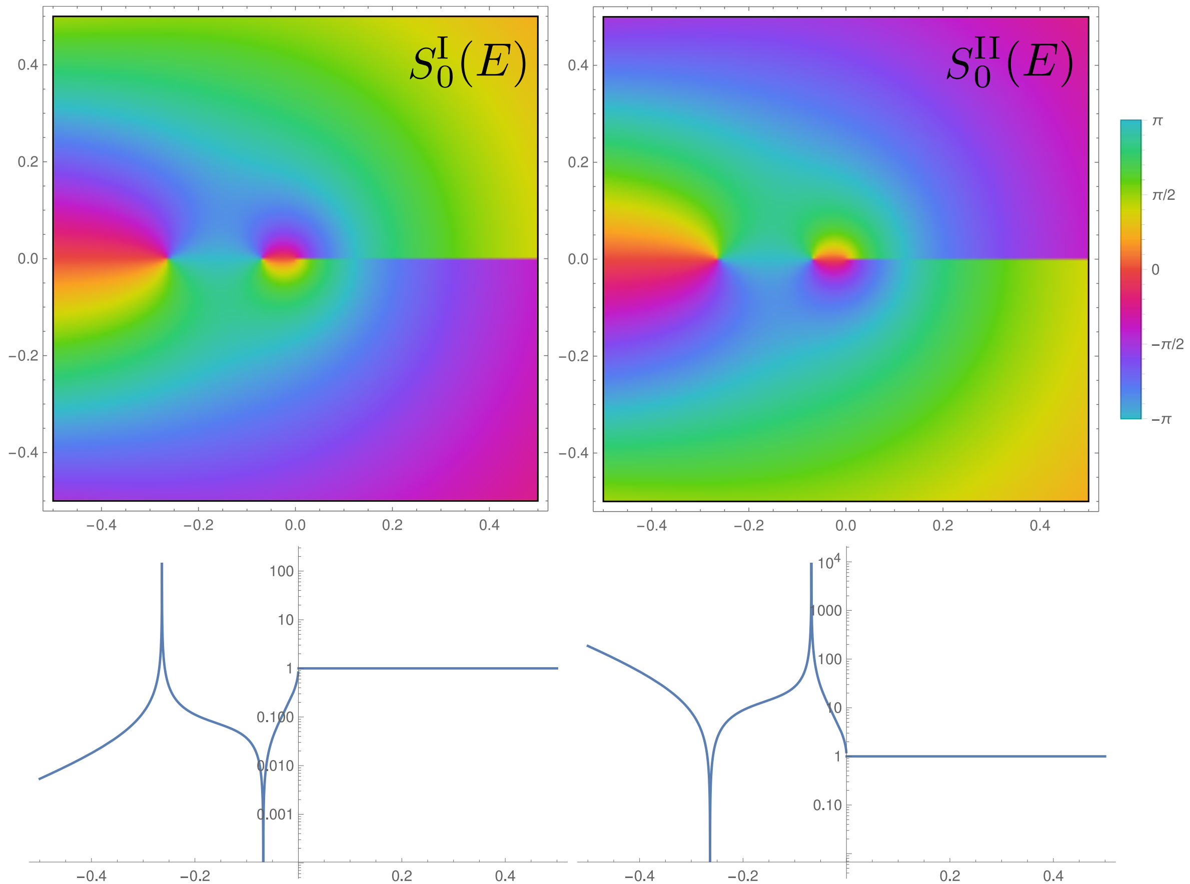
Leonid 2, via Wikimedia commons

Analytic structure of the S-matrix

- from the square-root structure it follows that the two sheets of the S-matrix as a function E correspond to the upper and lower half planes as a function of k



Example



calculation by Nuwan Yapa

Riemann sheets of the T-matrix

- consider now the (partial-wave projected) Lippmann-Schwinger equation in momentum space:

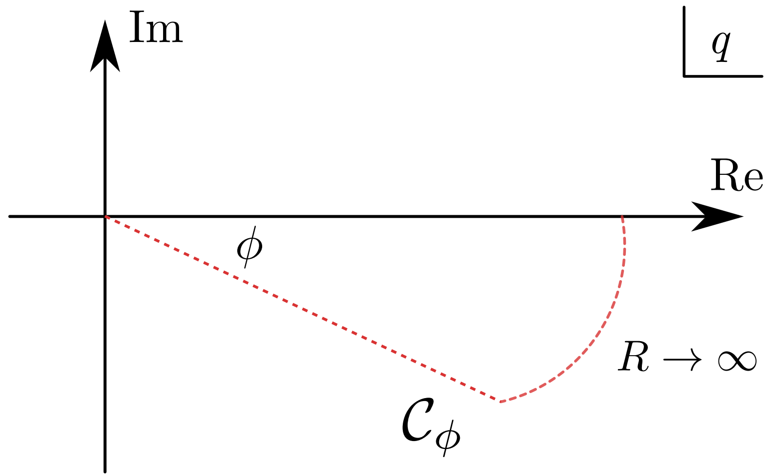
$$T(E;p,p') = V(p,p') + \int_0^\infty \frac{dq\,q^2}{2\pi^2} V(p,q) G_0(E;q) T(E;q,p') \tag{1}$$

- we have written this in full off-shell form, with the energy E a free parameter not associated with either p or p'
- just like the S-matrix, the T-matrix has two Riemann sheets, which in the following we denote by T^{I} and T^{II}, and Eq. (1) is the equation for $T = T^{\mathrm{I}}$
- that means, even if we choose E complex, we do not leave the first sheet

How then can we obtain T^{II}?

Contour rotation

- recall that the **scattering cut** connects the first and second Riemann sheets
  - it runs along the positive real axis
  - this is precisely where we integrate in the Lippmann-Schwinger equation: $\int_0^\infty \mathrm{d}q$
  - for scattering calculations, we use $\mathrm{i}\varepsilon \to 0$ to approach the upper rim of the cut
- let us now deform this integration contour by rotating it into the lower half plane



- the contribution from the arc can be neglected if both $V(p, q)$ and $T(E; q, p')$ fall off sufficiently fast for $q \to \infty$

Analytic continuation

- to rotate the contour in the first place, we need to assume of course that the potential is actually defined for complex momenta
  - for short-range local potentials this is just fine because the integral

$$V_l(p,k) = 4\pi \int_0^\infty \mathrm{d}r\, r^2 j_l(pr) V(r) j_l(kr)$$

  converges for all p and k
  - so-called separable potentials, i.e., potentials that factorize as

$$V(p,k) \sim g(p)g(k)$$

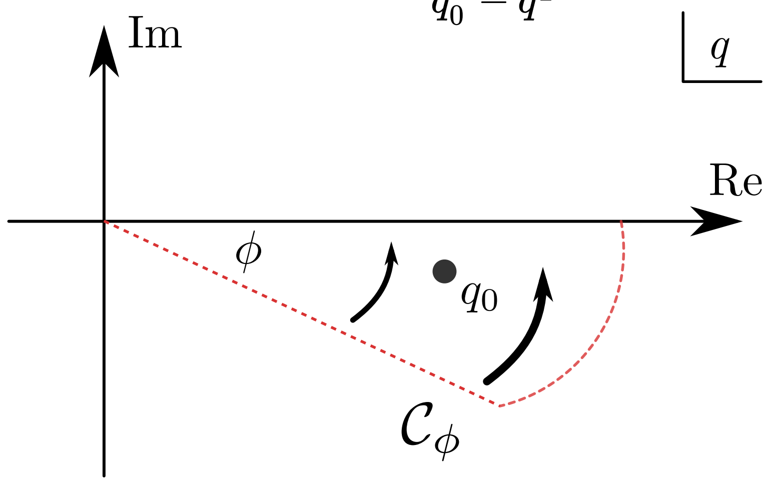
  are also no problem provided the "form factor" $g(p)$ is an analytic function of p
- after rotating the contour, we can pick E with $q_0 = \sqrt{2\mu E}$ such that $-\arg q_0 < \phi$ and write down the **Lippmann-Schwinger equation on the second sheet** as

$$T^{\mathrm{II}}(E;p,p') = V(p,p') + \int_{\mathcal{C}_\phi} \frac{dq\,q^2}{2\pi^2} V(p,q) G_0(E;q) T^{\mathrm{II}}(E;q,p') \tag{2}$$

Rotation reversed

- the contour-rotation method is strikingly simple, but it introduces the angle ϕ as an additional parameter in the calculation
- note now that the free Green's function for has a pole at $q = q_0 = \sqrt{2\mu E}$:

$$G_0(E; q) = \frac{2\mu}{q_0^2 - q^2} \tag{3}$$



- if we want to rotate the contour back to the real axis, we will sweep across this pole
- this means that we will **pick up a residue contribution**

for a more detailed discussion, see W. Glöckle, The Quantum Mechanical Few-Body Problem, Springer, 1983

Full circle

- let us retrace our steps so far:

1. without specifying the energy explicitly, we rotated the $\mathrm{d}q$ integral
2. we then chose the energy E in the accessible part of the second sheet
3. after fixing E, we rotate the integral back and pick up a residue

- this leads to the following equation:

$$\begin{aligned} T^{\mathrm{II}}(E;p,p') = V(p,p') &+ \int_0^\infty \frac{dq\,q^2}{2\pi^2} V(p,q) G_0(E;q) T^{\mathrm{II}}(E;q,p') \\ &- \frac{\mathrm{i}\mu q_0}{\pi} V(p,q_0) T^{\mathrm{II}}(E;q_0,p') \end{aligned} \tag{4}$$

- for the new amplitude $T^{\mathrm{II}}(E;q_0,p)$ we need a supplementary equation:

$$\begin{aligned} T^{\mathrm{II}}(E;q_0,p') = V(q_0,p') &+ \int_0^\infty \frac{dq\,q^2}{2\pi^2} V(q_0,q) G_0(E;q) T^{\mathrm{II}}(E;q,p') \\ &- \frac{\mathrm{i}\mu q_0}{\pi} V(q_0,q_0) T^{\mathrm{II}}(E;q_0,p') \end{aligned} \tag{5}$$

Second-sheet kernel

- in numerical calculations, where we discretize the $\mathrm{d}q$ integral, we can combine the two equations (4) and (5) by adding q_0 as an extra mesh point
- this is similar to our numerical treatment of the principal-value integral that we encountered for scattering calculations
- a yet simpler equation can be obtained by eliminating $T^{\mathrm{II}}(E; q_0, p')$ explicitly:

$$T^{\mathrm{II}}(E;p,p') = \tilde{V}(q_0;p,p') + \int_0^\infty \frac{dq\,q^2}{2\pi^2}\tilde{V}(q_0;p,q)G_0(E;q)T^{\mathrm{II}}(E;q,p')\,, \quad (6)$$

with

$$\tilde{V}(q_0;p,p') = V(p,p') - V(p,q_0)\frac{\mathrm{i}\mu q_0/\pi}{1+\mathrm{i}\mu q_0 V(q_0,q_0)/\pi}V(q_0,p') \quad (7)$$

- this **modified kernel for the second sheet** allows us to search for virtual states and resonances
- note that in all these equations, we have $q_0 = \sqrt{2\mu E}$

Second-sheet S-matrix poles

- in order to actually search for virtual states and resonances, we need to identify poles of the S-matrix on the second sheet
- as for bound states, the poles actually are poles of the T-matrix
- to find these poles, we **proceed exactly as we did for bound states**
- assuming the existence of simple pole at energy E^*, the second-sheet T-matrix factorizes at the pole position:

$$T^{\mathrm{II}}(E) \sim \frac{|R\rangle\langle R|}{E - E^*} \;\text{ for }\; E \to E^* \tag{8}$$

- we use $R(p) = \langle p|R\rangle$ here to denote the vertex function
- inserting this into the second-sheet Lippmann-Schwinger equation (6) yields the homogeneous equation

$$R(p) = \int \frac{\mathrm{d}q\, q^2}{2\pi^2} \tilde{V}(q_0; p, q) G_0(E^*; q) R(q) \,, \tag{9}$$

where now $q_0 = \sqrt{2\mu E^*}$